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Permeable cracks between two dissimilar piezoelectric materials

H.G. Beom *

Department of Mechanical Engineering, College of Engineering, Chonnam National University, 300 Yongbong-dong, Kwangju 500-757, South Korea

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Abstract

An interfacial crack with electrically permeable surfaces between two dissimilar piezoelectric ceramics under electromechanical loading is investigated. An exact expression for singular stress and electric fields near the tip of a permeable crack between two dissimilar anisotropic piezoelectric media are obtained. The interfacial crack-tip fields are shown to consist of both an inverse square root singularity and a pair of oscillatory singularities. It is found that the singular fields near the permeable interfacial crack tip are uniquely characterized by the real valued stress intensity factors proposed in this paper. The energy release rate is obtained in terms of the stress intensity factors. The exact solution of stress and electric fields for a finite interfacial crack problem is also derived.

Keywords: Piezoelectric ceramics; Interfacial crack; Crack-tip fields; Energy release rate

1. Introduction

Piezoelectric ceramics have potential applications in sensors, actuators and electromechanical devices due to their inherent coupled electromechanical behavior. Piezoelectric ceramics are susceptible to damage in the form of cracks due to their brittleness. Defects such as cracks are known to significantly influence the strength of piezoelectrics. In order to understand fracture behavior of piezoelectric materials under electromechanical loading, it is of great importance to investigate problems of cracks. Extensive studies on the subject of cracks have been carried out by many researchers. A summary of the results by previous works can be found in Zhang et al. (2002).

Studies on the fracture of bimaterials have been performed for an interface crack between two dissimilar piezoelectric materials under electromechanical loading. Considerable effort has been made to explore the structure of crack tip singular fields. Electrically impermeable, conductive or permeable conditions on the interfacial crack surfaces have been employed in the literature to study fracture behavior of piezoelectric

* Tel.: +82-62-530-1682; fax: +82-62-530-1689.
E-mail address: hgbeom@chonnam.ac.kr (H.G. Beom).

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bimaterials. Suo et al. (1992) solved the problem of an insulating crack between dissimilar anisotropic piezoelectric media. Their result showed that the interfacial insulating crack tip field consists of both a pair of oscillatory singularities and a pair of nonoscillatory singularities. Subsequently, Beom and Atluri (1996) derived a complete form of stress and electric displacement fields in the vicinity of the tip of an insulating crack between two dissimilar anisotropic piezoelectric media, and proposed new definitions of real-valued stress and electric displacement intensity factors for the interfacial crack. Recently, Beom and Atluri (2002) investigated the problem of a conducting crack on an interface between dissimilar anisotropic piezoelectric media. They discovered a new type of singularities around conducting interface crack tips, which consists of two pairs of oscillatory singularities. On the other hand, the permeable model has been employed to analyze an interfacial crack between two dissimilar piezoelectric materials (Wang and Han, 1999; Gao and Wang, 2000, 2001). Gao and Wang (2000, 2001) obtained the result in which the structure of singular fields near the permeable interfacial crack tip is the same as that near the impermeable interfacial crack tip. They also found that field intensity factors of a permeable crack can be obtained from the corresponding value of an impermeable crack.

It is the purpose of this study to investigate the problem of an interfacial crack on the interface between two dissimilar piezoelectric media. The crack surfaces are assumed to be electrically permeable. The problem is formulated using the complex representation. The closed form of the singular crack tip fields for the interface crack between dissimilar piezoelectric materials is derived here using an analysis based on analytic functions. A stress singularity type around the interface crack tip is discovered, which is a contrast to the result by previous works. A definition of real-valued stress intensity factors is proposed, and the energy release rate is obtained in terms of the stress intensity factors. A closed form of the solution for a finite crack on the interface between dissimilar anisotropic piezoelectric media is also derived.

2. Formulation

Let us consider a generalized two-dimensional deformation of a linear anisotropic piezoelectric solid. The three components of displacement and the electric potential are assumed to depend only on the in-plane Cartesian coordinate, x_1 and x_2. According to Barnett and Lothe (1975), a general solution for the displacement and electric field that satisfies the equations of equilibrium, and the corresponding stress and electric displacement components can be expressed in terms of four analytic functions as

$$\begin{aligned} v_J &= 2\,\mathrm{Re}\left[\sum_{K=1}^{4} A_{JK} f_K(z_K)\right], \\ \Sigma_{1J} &= -2\,\mathrm{Re}\left[\sum_{K=1}^{4} B_{JK} p_K f_K'(z_K)\right], \\ \Sigma_{2J} &= 2\,\mathrm{Re}\left[\sum_{K=1}^{4} B_{JK} f_K'(z_K)\right]. \end{aligned} \tag{1}$$

Here $v_j = u_j$ ($j = 1,2,3$), $v_4 = \phi$, $\Sigma_{ij} = \sigma_{ij}$ ($i = 1,2$; $j = 1,2,3$), and $\Sigma_{i4} = D_i$ ($i = 1,2$), where u_j, ϕ, σ_{ij} and D_i are the displacement, the electric potential, the stress and the electric displacement, respectively. Re denotes the real part, prime (′) represents the derivative with respect to the associated arguments, and $f_K(z_K)$ are analytic in their arguments, $z_K = x_1 + p_K x_2$; and p_K are four distinct complex numbers with positive imaginary. The result of general solution enables us to formulate a boundary value problem in terms of the complex potentials. The solution to a problem of a piezoelectric material is reduced to finding the functions $f_K(z_K)$, which satisfy the boundary conditions of the problem. For convenience, the one-complex-variable approach introduced by Suo (1990) is employed to present our solutions through the vector function, $\mathbf{f}(z)$, defined as

$$\mathbf{f}(z) = (\, f_1(z) \quad f_2(z) \quad f_3(z) \quad f_4(z)\,)^{\mathrm{T}}. \tag{2}$$

Here the argument has the generic form $z = x_1 + px_2$ in which $\mathrm{Im}\, p > 0$. Im denotes the imaginary part. Superscript T indicates the transpose and boldfaced symbols represent vectors or matrices in this paper. Once the solution of $\mathbf{f}(z)$ is obtained for a given boundary value problem, a replacement of z_1, z_2, z_3 or z_4 should be made for each component function to calculate field quantities.

A bimaterial matrix is defined as

$$\mathbf{H} = \mathrm{i}\mathbf{A}^{(1)}\mathbf{B}^{(1)-1} + \overline{\mathrm{i}\mathbf{A}^{(2)}\mathbf{B}^{(2)-1}}, \tag{3}$$

where superscripts 1 and 2 in parentheses indicate that the quantities are for the materials 1 and 2 composing the bimaterial, respectively, and overbar (–) denotes the complex conjugate. Lothe and Barnett (1976) showed that a 3×3 matrix with the components H_{ij} $(i,j = 1,2,3)$ is positive definite and $H_{44} < 0$. $\mathbf{H}$ is also Hermitian (Suo et al., 1992). We introduce a 3×3 bimaterial matrix $\widehat{\mathbf{H}}$, which will be used subsequently in this paper, defined as

$$\widehat{H}_{ij} = H_{ij} - \frac{1}{H_{44}} H_{i4}H_{4j}, \tag{4}$$

It can be easily shown from (4) that

$$\widehat{H}_{ij}^{-1} = H_{ij}^{-1}, \tag{5}$$

Making use of the properties of $\mathbf{H}$, it can be shown that $\widehat{\mathbf{H}}$ is a positive definite Hermitian matrix. Thus, the bimaterial matrix $\widehat{\mathbf{H}}$ can be written as

$$\widehat{\mathbf{H}} = \boldsymbol{\gamma} - \mathrm{i}\boldsymbol{\omega}, \tag{6}$$

where $\boldsymbol{\gamma}$ is the real symmetric positive-definite matrix and $\boldsymbol{\omega}$ is the real anti-symmetric matrix:

$$\boldsymbol{\beta} = \boldsymbol{\gamma}^{-1}\boldsymbol{\omega}. \tag{7}$$

Here the bimaterial matrix $\boldsymbol{\beta}$ can be interpreted as one of generalized Dundurs parameters for an anisotropic piezoelectric bimaterial. Various versions of generalized Dundurs parameters have been proposed for a piezoelectric bimaterial by Beom and Atluri (1996, 2002) and for an anisotropic elastic bimaterial by Beom and Atluri (1995). Since $\boldsymbol{\gamma}$ is symmetric and $\boldsymbol{\omega}$ is anti-symmetric, it can be shown that the bimaterial matrix $\boldsymbol{\beta}$ satisfies the following relations:

$$\mathrm{tr}(\boldsymbol{\beta}) = 0, \qquad \mathrm{tr}(\boldsymbol{\beta}^2) \leqslant 0, \qquad \|\boldsymbol{\beta}\| = 0, \tag{8}$$

where tr represents the trace of a matrix and $\|\cdot\|$ denotes the determinant of a matrix.

3. Singular crack tip field

Consider a permeable crack between two piezoelectric ceramics. We are specifically interested in singular crack tip fields. In order to obtain a singular solution near the interface crack tip, we consider a semi-infinite crack on the interface between two dissimilar, linear piezoelectric materials with material 1 above and material 2 below as shown in Fig. 1. The crack lies along the negative x_1 axis, and the crack is traction-free and electrically permeable.

Continuity of tractions and normal electric displacement across the entire interface, both the bonded and cracked portions requires that

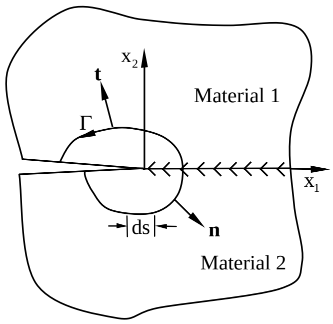

Fig. 1. Semi-infinite interfacial crack.

$$\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(x_1) - \overline{\mathbf{B}}^{(2)}\mathbf{f}'^{(2)}(x_1) = \mathbf{B}^{(2)}\mathbf{f}'^{(2)}(x_1) - \overline{\mathbf{B}}^{(1)}\overline{\mathbf{f}}'^{(1)}(x_1). \tag{9}$$

By the standard analytic continuation arguments, we obtain from (9) that

$$\mathbf{B}^{(2)}\mathbf{f}'^{(2)}(z) = \overline{\mathbf{B}}^{(1)}\overline{\mathbf{f}}'^{(1)}(z). \tag{10}$$

Eqs. (1) and (10) give

$$\mathrm{i}\Delta\mathbf{v}_{,1} = \mathbf{H}\left[\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(x_1) - \mathbf{H}^{-1}\overline{\mathbf{H}}\mathbf{B}^{(2)}\mathbf{f}'^{(2)}(x_1)\right]. \tag{11}$$

Here $\Delta\mathbf{v} = \mathbf{v}(x_1, 0^+) - \mathbf{v}(x_1, 0^-)$. With the same arguments, the continuity of the displacement and electric potential across the bonded interface requires that a function defined as

$$\mathbf{g}'(z) = \begin{cases} \mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z), \\ \mathbf{H}^{-1}\overline{\mathbf{H}}\mathbf{B}^{(2)}\mathbf{f}'^{(2)}(z) \end{cases} \tag{12}$$

is analytic everywhere in the region surrounding the crack tip except on the crack line. Furthermore, the permeable condition on x_1-axis leads to

$$H_{4J}g_J'^{+}(x_1) = H_{4J}g_J'^{-}(x_1). \tag{13}$$

In this paper the repetition of an index in a term denotes a summation with respect to that index over its range 1–3 for a lowercase script and 1–4 for an uppercase script, unless indicated otherwise. By the standard analytic continuation arguments, we see from (13) for the singular solution that

$$H_{4J}g_J'(z) = 0. \tag{14}$$

Solving (14) for $g_4'(z)$, we get

$$g_4'(z) = -\frac{H_{4j}}{H_{44}}g_j'(z), \tag{15}$$

The traction-free condition on the surface of the crack can be shown to lead to a homogeneous Hilbert problem

$$\hat{\mathbf{g}}'^{+}(x_1) + \overline{\widehat{\mathbf{H}}}^{-1}\widehat{\mathbf{H}}\hat{\mathbf{g}}'^{-}(x_1) = 0, \quad x_1 < 0, \tag{16}$$

where $\hat{\mathbf{g}}'(z) = \left(g_1'(z) \quad g_2'(z) \quad g_3'(z) \right)^{\mathrm{T}}$. Substituting (6) and (7) into (16), it is found that

$$\hat{\mathbf{g}}'^{+}(x_1) + (\mathbf{I} + \mathrm{i}\boldsymbol{\beta})^{-1}(\mathbf{I} - \mathrm{i}\boldsymbol{\beta})\hat{\mathbf{g}}'^{-}(x_1) = 0, \tag{17}$$

where $\mathbf{I}$ is the identity matrix. The singular solution of (17) for $\hat{\mathbf{g}}'(z)$ can be obtained from a well-known solution in the linear fracture mechanics of crack in an anisotropic elastic material due to Wu (1990), Qu and Li (1991) and Beom and Atluri (1995). A similar procedure permits us to obtain the singular solution from the anisotropic elastic solutions. The singular solution of (17) for $\hat{\mathbf{g}}'(z)$ is given by (see Appendix A for details)

$$\hat{\mathbf{g}}'(z) = \frac{1}{2\sqrt{2\pi z}}(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})\mathbf{Y}(z^{\mathrm{i}\varepsilon})\mathbf{k}, \tag{18}$$

in which

$$\varepsilon = \frac{1}{2\pi}\ln\frac{1+\lambda}{1-\lambda}, \quad \lambda = \left[-\frac{1}{2}\mathrm{tr}(\boldsymbol{\beta}^2)\right]^{1/2}, \tag{19}$$

where $\mathbf{k} = (k_1 \quad k_2 \quad k_3)^{\mathrm{T}}$ is a real constant vector. It is seen from (8) and (19) that ε is a real number depending on the real bimaterial matrix $\boldsymbol{\beta}$. The matrix function $\mathbf{Y}(\zeta(z))$ is defined as $\mathbf{Y}(\zeta(z)) \equiv \mathbf{Y}(\zeta(z), \overline{\zeta}(z))$ in which $\zeta(z)$ is an arbitrary function of z and $\mathbf{Y}(\zeta_1, \zeta_2)$ is expressed explicitly in terms of the real bimaterial matrix $\boldsymbol{\beta}$ as

$$\mathbf{Y}(\zeta_1, \zeta_2) = \mathbf{I} + \frac{\mathrm{i}}{2\lambda}(\zeta_1 - \zeta_2)\boldsymbol{\beta} + \frac{1}{\lambda^2}\left[1 - \frac{1}{2}(\zeta_1 + \zeta_2)\right]\boldsymbol{\beta}^2. \tag{20}$$

The matrix function $\mathbf{Y}(\zeta_1, \zeta_2)$ given by (20) can be shown to have the following properties:

$$\mathbf{Y}(1,1) = \mathbf{I}, \qquad \mathbf{Y}(\zeta_1, \zeta_2)\mathbf{Y}(\xi_1, \xi_2) = \mathbf{Y}(\zeta_1\xi_1, \zeta_2\xi_2). \tag{21}$$

Integrating (18), we have

$$\hat{\mathbf{g}}(z) = \sqrt{\frac{z}{2\pi}}(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})\mathbf{Y}\left(\frac{z^{\mathrm{i}\varepsilon}}{1 + 2\mathrm{i}\varepsilon}\right)\mathbf{k}, \tag{22}$$

Since $\hat{\mathbf{g}}'(z)$ is determined as above, $\mathbf{f}'^{(1)}(z)$ and $\mathbf{f}'^{(2)}(z)$ can be obtained from (10), (12) and (18). Thus, we have

$$\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z) = \frac{1}{2\sqrt{2\pi z}}\mathbf{Q}(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})\mathbf{Y}(z^{\mathrm{i}\varepsilon})\mathbf{k}, \qquad \mathbf{B}^{(2)}\mathbf{f}'^{(2)}(z) = \frac{1}{2\sqrt{2\pi z}}\overline{\mathbf{Q}}(\mathbf{I} - \mathrm{i}\boldsymbol{\beta})\mathbf{Y}(z^{\mathrm{i}\varepsilon})\mathbf{k}, \tag{23}$$

where $\mathbf{Q}$ is the 4×3 matrix with the components $Q_{ij} = \delta_{ij}$ and $Q_{4j} = -H_{4j}/H_{44}$. It is noted that the permeable interfacial crack tip fields have the inverse square root singularity and a pair of oscillatory singularities, while the insulating interfacial crack tip fields consist of a pair of oscillatory singularities and a pair of nonoscillatory singularities. Thus, the structure of singular fields near the permeable interfacial crack tip differs from that near the impermeable interfacial crack tip, in contrast to the recent conclusion by Gao and Wang (2000, 2001). The singular solution for the special case in which the bimaterial continuum degenerates to be a homogeneous one reduces to

$$\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z) = \mathbf{B}^{(2)}\mathbf{f}'^{(2)}(z) = \frac{1}{2\sqrt{2\pi z}}\mathbf{Q}\mathbf{k}, \tag{24}$$

The singular stress field along the bonded interface near the crack tip is given by

$$\boldsymbol{\tau}(x_1) = \frac{1}{\sqrt{2\pi x_1}}\mathbf{Y}(x_1^{\mathrm{i}\varepsilon}, x_1^{-\mathrm{i}\varepsilon})\mathbf{k}, \tag{25}$$

where $\boldsymbol{\tau} = (\sigma_{21} \quad \sigma_{22} \quad \sigma_{23})^{\mathrm{T}}$. Thus, the vector of stress intensity factor which uniquely characterizes the singular field can be defined through the equation

$$\mathbf{k} = \lim_{x_1 \to 0^+} \sqrt{2\pi x_1} \mathbf{Y}(x_1^{-i\varepsilon}, x_1^{i\varepsilon}) \boldsymbol{\tau}(x_1), \tag{26}$$

where $\mathbf{k} = (K_2 \quad K_1 \quad K_3)^{\mathrm{T}}$. Since $\mathbf{Y}(x_1^{-i\varepsilon}, x_1^{i\varepsilon})$ and $\boldsymbol{\tau}(x_1)$ are real, $\mathbf{k}$ is real. A stress intensity factor with the same dimension of classical stress intensity factor, denoted by $\mathbf{k}_l^*$ also can be defined based on the characteristic length l as suggested by Wu (1990) and Qu and Li (1991) for the anisotropic elastic bimaterial case. $\hat{\mathbf{k}}_l^*$ is related to $\mathbf{k}$ by $\mathbf{k}_l^* = \mathbf{Y}(l^{i\varepsilon}, l^{-i\varepsilon})\mathbf{k}$. It is noted that the stress intensity factor $\mathbf{k}$ given in (26) for the piezoelectric bimaterial recovers the classical intensity factor $(K_{II} \quad K_I \quad K_{III})^{\mathrm{T}}$ as the bimaterial continuum degenerates to be a homogeneous one.

The electric displacement at the bonded interface ($x_1 > 0$) is given by

$$D_2(x_1, 0) = \frac{1}{\sqrt{2\pi x_1}} \mathrm{Re}[\mathbf{h}^{\mathrm{T}}(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})] \mathbf{Y}(x_1^{i\varepsilon}, x_1^{-i\varepsilon}) \mathbf{k}, \tag{27}$$

where $\mathbf{h} = -\frac{1}{H_{44}}(H_{41} \quad H_{42} \quad H_{43})^{\mathrm{T}}$. It is noted that the singular electric displacement is characterized by the stress intensity factors, and has the oscillatory singularity. For the special case of a homogeneous piezoelectric material, the intensity factor of electric displacement defined as $K_D = \lim_{x_1 \to 0^+} \sqrt{2\pi x_1} D_2(x_1)$ is given from (27) as

$$K_D = -\frac{H_{4j}}{H_{44}} k_j. \tag{28}$$

The electric displacement on the crack surfaces ($x_1 < 0$) is given by

$$D_2^+(x_1, 0) = D_2^-(x_1, 0) = \frac{1}{\sqrt{-2\pi x_1}} \mathrm{Im}[\mathbf{h}^{\mathrm{T}}] \mathbf{Y}(1/\cosh\pi\varepsilon) \mathbf{Y}((-x_1)^{i\varepsilon}, (-x_1)^{-i\varepsilon}) \mathbf{k}. \tag{29}$$

In deriving (29), the following relation has been used

$$(\mathbf{I} + \mathrm{i}\boldsymbol{\beta}) \mathbf{Y}(\mathrm{e}^{-\pi\varepsilon}, \mathrm{e}^{\pi\varepsilon}) = (\mathbf{I} - \mathrm{i}\boldsymbol{\beta}) \mathbf{Y}(\mathrm{e}^{\pi\varepsilon}, \mathrm{e}^{-\pi\varepsilon}) = \mathbf{Y}(1/\cosh\pi\varepsilon). \tag{30}$$

It is easily seen from (29) that when the bimaterial continuum degenerates to be a homogeneous one, $D_2^+(x_1, 0) = D_2^-(x_1, 0) = 0$ on the crack surfaces. This implies that the structure of singular fields for a permeable crack in a homogeneous piezoelectric material is identical to that for the special case of an impermeable crack with the condition (28).

The displacement jump at distance r behind of the crack tip, calculated from (1) and (23), is given by

$$\Delta\mathbf{u}(r) = \sqrt{\frac{2r}{\pi}} \mathrm{Re}(\widehat{\mathbf{H}}) \mathbf{Y}\left(\frac{r^{i\varepsilon}}{\cosh\pi\varepsilon(1 + 2i\varepsilon)}, \frac{r^{-i\varepsilon}}{\cosh\pi\varepsilon(1 - 2i\varepsilon)} \right) \mathbf{k}, \tag{31}$$

where $\Delta\mathbf{u}(r) = \mathbf{u}(x_1, 0^+) - \mathbf{u}(x_1, 0^-)$. In deriving (31), the following relation together with (30) has been used

$$(HQ)_{jk} = \widehat{H}_{jk}. \tag{32}$$

4. Energy release rate

The J integral for a linear piezoelectric medium, which has the physical meaning of energy release rate due to crack extension, is defined by (Cherepanov, 1979; Pak, 1990)

$$J\{\mathbf{v}; \Gamma\} = \int_{\Gamma} (W n_1 - t_J v_{J,1}) \, \mathrm{d}s. \tag{33}$$

Here W is the electric enthalpy density, given by $W = \frac{1}{2}\Sigma_{iJ}v_{J,i}$. n_i is the unit outward normal vector and t_J is the surface traction, given by $t_J = n_i\Sigma_{iJ}$. Γ is a path connecting any two points on opposite sides of the crack surface and enclosing the crack tip and ds is an element of arc length along Γ as shown in Fig. 1. It is well known that the generalized J integral is independence of any path Γ.

According to Beom and Atluri (1996), the J integral is written in the complex form, for an anisotropic piezoelectric solid, as

$$J = \mathrm{Re}\left[\sum_{J=1}^{4}\int_{\Gamma}\{f_J'(z_J)\}^2\,\mathrm{d}z_J\right], \tag{34}$$

where $\mathbf{f}(z)$ is the normalized function associated with the matrices $\mathbf{A}$ and $\mathbf{B}$ normalized according to $2A_{IJ}B_{IJ} = 1$ (no sum on J). Since the complete general solutions for the near tip fields are determined as shown in the previous section, the relation between the J integral and the intensity factors can be derived through the complex formula of the J integral. The J integral is evaluated with near tip fields given by (23), resulting in

$$J = \tfrac{1}{4}\mathbf{k}^{\mathrm{T}}\,\mathrm{Re}(\widehat{\mathbf{H}})(\mathbf{I}+\boldsymbol{\beta}^2)\mathbf{k}. \tag{35}$$

In obtaining (35), the following relations have been used:

$$\begin{aligned}&\mathbf{Y}(z^{\mathrm{i}\varepsilon}) = \overline{\mathbf{Y}(\bar{z}^{\mathrm{i}\varepsilon})}, \qquad \mathbf{Y}^{\mathrm{T}}(z^{\mathrm{i}\varepsilon})\,\mathrm{Re}(\widehat{\mathbf{H}})(\mathbf{I}+\boldsymbol{\beta}^2)\mathbf{Y}(z^{\mathrm{i}\varepsilon}) = \mathrm{Re}(\widehat{\mathbf{H}})(\mathbf{I}+\boldsymbol{\beta}^2),\\ &(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})^{\mathrm{T}}\,\mathrm{Re}(\widehat{\mathbf{H}})(\mathbf{I}+\mathrm{i}\boldsymbol{\beta}) = \mathrm{Re}(\widehat{\mathbf{H}})(\mathbf{I}+\boldsymbol{\beta}^2), \qquad \mathbf{Q}^{\mathrm{T}}\,\mathrm{Re}(\mathbf{H})\mathbf{Q} = \mathrm{Re}(\widehat{\mathbf{H}}).\end{aligned} \tag{36}$$

The energy release rate can be calculated by using the crack- closure energy. Thus, the energy release rate G is given by

$$G = \lim_{\Delta a\to 0}\frac{1}{2\Delta a}\int_0^{\Delta a}\boldsymbol{\tau}^{\mathrm{T}}(r)\Delta\mathbf{u}(\Delta a - r)\,\mathrm{d}r, \tag{37}$$

where

$$\begin{aligned}&\boldsymbol{\tau}(r) = \frac{1}{\sqrt{2\pi r}}\mathbf{Y}(r^{\mathrm{i}\varepsilon}, r^{-\mathrm{i}\varepsilon})\mathbf{k},\\ &\Delta\mathbf{u}(\Delta a - r) = \sqrt{\frac{2(\Delta a - r)}{\pi}}\mathrm{Re}(\widehat{\mathbf{H}})\mathbf{Y}\left(\frac{(\Delta a - r)^{\mathrm{i}\varepsilon}}{\cosh\pi\varepsilon(1+2\mathrm{i}\varepsilon)}, \frac{(\Delta a - r)^{-\mathrm{i}\varepsilon}}{\cosh\pi\varepsilon(1-2\mathrm{i}\varepsilon)}\right)\mathbf{k}.\end{aligned} \tag{38}$$

In obtaining (38), the condition $\Delta v_4(\Delta a - r) = 0$ has been used. Making use of the following relations:

$$\mathbf{Y}^{\mathrm{T}}(r^{\mathrm{i}\varepsilon}, r^{-\mathrm{i}\varepsilon})\,\mathrm{Re}(\widehat{\mathbf{H}}) = \mathrm{Re}(\widehat{\mathbf{H}})\mathbf{Y}^{\mathrm{T}}(r^{-\mathrm{i}\varepsilon}, r^{\mathrm{i}\varepsilon}), \qquad \mathbf{Y}(1/\cos sh^2\pi\varepsilon) = \mathbf{I}+\boldsymbol{\beta}^2, \tag{39}$$

it can be shown that

$$G = \tfrac{1}{4}\mathbf{k}^{\mathrm{T}}\,\mathrm{Re}(\widehat{\mathbf{H}})(\mathbf{I}+\boldsymbol{\beta}^2)\mathbf{k}. \tag{40}$$

The J integral in (34), as computed directly through the complex formula of the J integral coincides with the crack-closure energy G. For the special case in which the bimaterial continuum degenerates to be a homogeneous one, (35) reduces to

$$J = \tfrac{1}{4}\mathbf{k}^{\mathrm{T}}\widehat{\mathbf{H}}\mathbf{k}. \tag{41}$$

This is identical to the well-known result for the homogeneous case (Wang and Han, 1999).

5. Finite interfacial crack

Consider a finite crack, in the interval $(-a, a)$, on the permeable interface between dissimilar piezoelectric media. Due to the linearity of the problem, the solution of the problem under electromechanical loading at infinity can be decomposed into two solutions corresponding to the solution of the uncracked body subject to electromechanical tractions at infinity and the solution of the cracked body subject to tractions on the crack surfaces. Our attention focuses on the problem of a piezoelectric bimaterial subject to the tractions acting on the crack surfaces as shown in Fig. 2. Tractions $\hat{\mathbf{t}}^{+}(x_1) = \hat{\mathbf{t}}^{0}(x_1)$ and $\hat{\mathbf{t}}^{-}(x_1) = -\hat{\mathbf{t}}^{0}(x_1)$ are applied on the upper and lower surfaces of the crack, respectively. Here $\hat{\mathbf{t}} = (\, t_1 \quad t_2 \quad t_3 \,)^{\mathrm{T}}$. The solution procedure is similar to the case of the semi-infinite crack. For a finite crack in interval $(-a, a)$, the boundary condition on the crack surfaces leads to the following Hilbert problem for the determination of $\mathbf{f}'^{(1)}(z)$:

$$(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})\mathbf{y}^{+}(x_1) + (\mathbf{I} - \mathrm{i}\boldsymbol{\beta})\mathbf{y}^{-}(x_1) = -\hat{\mathbf{t}}^{0}, \quad -a < x_1 < a. \tag{42}$$

where $\mathbf{y}(z) = (\mathbf{I} + \mathrm{i}\boldsymbol{\beta})^{-1}\hat{\mathbf{g}}'(z)$. A homogeneous solution $\mathbf{X}(z)$ which satisfies the homogeneous Hilbert problem

$$(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})\mathbf{X}^{+}(x_1) + (\mathbf{I} - \mathrm{i}\boldsymbol{\beta})\mathbf{X}^{-}(x_1) = 0, \quad -a < x_1 < a, \tag{43}$$

may be written as

$$\mathbf{X}(z) = \frac{1}{\sqrt{z^2 - a^2}}\mathbf{Y}\left(\left[\frac{z-a}{z+a}\right]^{\mathrm{i}\varepsilon}\right). \tag{44}$$

From (42) and (43), we find

$$\mathbf{y}(z) = \frac{1}{2\pi\mathrm{i}}\mathbf{X}(z)\int_{-a}^{a}\frac{1}{z-\xi}[(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})\mathbf{X}^{+}(\xi)]^{-1}\hat{\mathbf{t}}^{0}\,\mathrm{d}\xi. \tag{45}$$

Using (44) and (45), it can be shown that a solution of $\mathbf{f}'(z)$ is given by

$$\begin{aligned}
\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z) &= \frac{1}{2\pi\sqrt{z^2 - a^2}}\mathbf{QY}\left(\left[\frac{z-a}{z+a}\right]^{\mathrm{i}\varepsilon}\right)\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}}{z-\xi}\mathbf{Y}(\zeta_0^{-\mathrm{i}\varepsilon}, \zeta_0^{\mathrm{i}\varepsilon})\hat{\mathbf{t}}^{0}\,\mathrm{d}\xi, \\
\mathbf{B}^{(2)}\mathbf{f}'^{(2)}(z) &= \frac{1}{2\pi\sqrt{z^2 - a^2}}\overline{\mathbf{Q}}\mathbf{Y}\left(\left[\frac{z-a}{z+a}\right]^{\mathrm{i}\varepsilon}\right)\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}}{z-\xi}\overline{\mathbf{Y}(\zeta_0^{-\mathrm{i}\varepsilon}, \zeta_0^{\mathrm{i}\varepsilon})}\hat{\mathbf{t}}^{0}\,\mathrm{d}\xi,
\end{aligned} \tag{46}$$

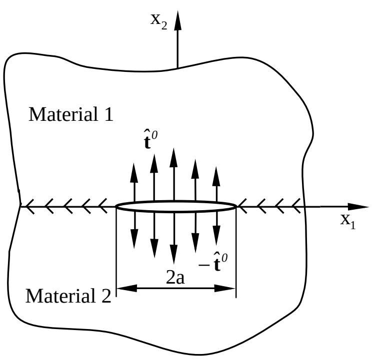

Fig. 2. Finite interfacial crack with mechanical crack facing loading.

where $\zeta_0 = \frac{a-\xi}{a+\xi}\mathrm{e}^{\mathrm{i}\pi}$. The stress intensity factor is evaluated by using (26) and (46), which results in

$$\mathbf{k} = \frac{1}{\sqrt{\pi a}}\int_{-a}^{a}\mathbf{Y}((2a\zeta_0)^{-\mathrm{i}\varepsilon}, (2a\zeta_0)^{\mathrm{i}\varepsilon})(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})^{-1}\hat{\mathbf{t}}^0\sqrt{\frac{a+\xi}{a-\xi}}\mathrm{d}\xi. \tag{47}$$

For the case in which $\hat{\mathbf{t}}^0$ is a constant vector, (46) reduces to

$$\begin{aligned}\mathbf{B}^{(1)}\mathbf{f}'^{(1)}(z) &= \frac{1}{2}\mathbf{Q}(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})\left[\frac{z}{\sqrt{z^2-a^2}}\mathbf{Y}\left(\left[\frac{z-a}{z+a}\right]^{\mathrm{i}\varepsilon}\right)-\mathbf{I}\right]\hat{\mathbf{t}}^0,\\ \mathbf{B}^{(2)}\mathbf{f}'^{(2)}(z) &= \frac{1}{2}\overline{\mathbf{Q}}(\mathbf{I}-\mathrm{i}\boldsymbol{\beta})\left[\frac{z}{\sqrt{z^2-a^2}}\mathbf{Y}\left(\left[\frac{z-a}{z+a}\right]^{\mathrm{i}\varepsilon}\right)-\mathbf{I}\right]\hat{\mathbf{t}}^0.\end{aligned} \tag{48}$$

In the derivation of (48), the following relation has been used:

$$\int_{-a}^{a}\frac{\sqrt{a^2-\xi^2}}{z-\xi}\mathbf{Y}(\zeta_0^{-\mathrm{i}\varepsilon}, \zeta_0^{\mathrm{i}\varepsilon})\,\mathrm{d}\xi = \pi(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})\left\{z\mathbf{I}-\sqrt{z^2-a^2}\mathbf{Y}\left(\left[\frac{z-a}{z+a}\right]^{-\mathrm{i}\varepsilon}\right)\right\}. \tag{49}$$

The stress intensity factor for the special case is given by

$$\mathbf{k} = \sqrt{\pi a}\mathbf{Y}((2a)^{-\mathrm{i}\varepsilon}, (2a)^{\mathrm{i}\varepsilon})\hat{\mathbf{t}}^0. \tag{50}$$

It is noted that the stress intensity factors depend only on applied mechanical loading, which is consistent with the result obtained by Gao and Wang (2000). The electric displacement and the electric field on the crack surfaces ($|x_1| < a$) are evaluated from (1) and (48), which results in

$$\begin{aligned}D_2^+(x_1,0) &= D_2^-(x_1,0)\\ &= -\mathrm{Re}[\mathbf{h}^{\mathrm{T}}(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})]\hat{\mathbf{t}}^0 + \frac{x_1}{\sqrt{a^2-x_1^2}}\mathrm{Im}[\mathbf{h}^{\mathrm{T}}]\mathbf{Y}(1/\cosh\pi\varepsilon)\mathbf{Y}\left(\left(\frac{a-x_1}{a+x_1}\right)^{\mathrm{i}\varepsilon}, \left(\frac{a-x_1}{a+x_1}\right)^{-\mathrm{i}\varepsilon}\right)\hat{\mathbf{t}}^0,\\ E_1^+(x_1,0) &= E_1^-(x_1,0)\\ &= -\mathrm{Re}[\mathbf{h}^{*\mathrm{T}}(\mathbf{I}+\mathrm{i}\boldsymbol{\beta})]\hat{\mathbf{t}}^0 + \frac{x_1}{\sqrt{a^2-x_1^2}}\mathrm{Im}[\mathbf{h}^{*\mathrm{T}}]\mathbf{Y}(1/\cosh\pi\varepsilon)\mathbf{Y}\left(\left(\frac{a-x_1}{a+x_1}\right)^{\mathrm{i}\varepsilon}, \left(\frac{a-x_1}{a+x_1}\right)^{-\mathrm{i}\varepsilon}\right)\hat{\mathbf{t}}^0,\end{aligned} \tag{51}$$

where

$$h_j^* = -\tfrac{1}{2}[(A^{(1)}B^{(1)-1} + \overline{A^{(2)}B^{(2)-1}})Q]_{4j}. \tag{52}$$

In obtaining (51), (30) and the following relation:

$$[A^{(1)}B^{(1)-1}Q]_{4j} = [\overline{A^{(2)}B^{(2)-1}}Q]_{4j}, \tag{53}$$

has been used. It is noted that the electric displacement and the electric field on the crack surfaces are oscillatory singular near the crack tip. For the special case of a homogeneous piezoelectric material, (48) reduces to

$$\mathbf{B}\mathbf{f}'(z) = \frac{1}{2}\mathbf{Q}\left[\frac{z}{\sqrt{z^2-a^2}}-1\right]\hat{\mathbf{t}}^0. \tag{54}$$

The stress intensity factor for the special case of a homogeneous piezoelectric material is obtained from (50) as

$$\mathbf{k} = \sqrt{\pi a}\hat{\mathbf{t}}^0. \tag{55}$$

It is also easily seen from (51) that the electric displacement and the electric field on the crack surfaces ($|x_1| < a$) are of the forms

$$D_2^+(x_1, 0) = D_2^-(x_1, 0) = -\mathbf{h}^{\mathrm{T}}\hat{\mathbf{t}}^0, \qquad E_1^+(x_1, 0) = E_1^-(x_1, 0) = -\mathbf{h}^{*\mathrm{T}}\hat{\mathbf{t}}^0. \tag{56}$$

The result of the electric displacement on the crack surface is consistent with that obtained by Gao and Wang (2000).

6. Concluding remarks

A crack with electrically permeable surfaces on the interface between two dissimilar piezoelectric ceramics under electromechanical loading is considered. The analytic solution of the singular crack-tip fields for a permeable interfacial crack between dissimilar piezoelectric materials is obtained here using an analysis based on the complex function theory. It is shown that the singular fields near the permeable interfacial crack tip are uniquely characterized by the real valued stress intensity factors proposed in this paper. The permeable interfacial crack-tip fields consist of both an inverse square root singularity and a pair of oscillatory singularities. Thus, the structure of singular fields near the permeable interfacial crack tip is in general different from that near an impermeable interfacial crack tip, in contrast to the result by previous works. When the bimaterial continuum degenerates to be a homogeneous one, however, the structure of singular fields for a permeable crack reduces to that for an impermeable crack with different intensity factors. The energy release rate is obtained in terms of the stress intensity factors through the complex formula of the J integral, which coincides with that calculated by using the crack-closure energy. A closed form of the solution for a finite permeable crack on the interface between dissimilar piezoelectric media is also derived.

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Appendix A. Derivation of (18)

Introducing a vector function $\boldsymbol{\chi}(z)$ defined by

$$\boldsymbol{\chi}(z) = (\mathbf{I} + \mathrm{i}\boldsymbol{\beta})^{-1}\hat{\mathbf{g}}'(z), \tag{A.1}$$

Eq. (17) is rewritten as

$$(\mathbf{I} + \mathrm{i}\boldsymbol{\beta})\boldsymbol{\chi}^+(x_1) + (\mathbf{I} - \mathrm{i}\boldsymbol{\beta})\boldsymbol{\chi}^-(x_1) = 0, \quad -\infty < x_1 < 0, \tag{A.2}$$

A singular solution of (A.2) for $\boldsymbol{\chi}(z)$ can be found by considering functions of the form $\boldsymbol{\chi}(z) = z^{-1/2+\mathrm{i}\delta}\mathbf{v}$, where $\mathbf{v}$ is an eigenvector. Substitution of $\boldsymbol{\chi}(z) = z^{-1/2+\mathrm{i}\delta}\mathbf{v}$ into (A.2) yields

$$(\boldsymbol{\beta} + \mathrm{i}\eta\mathbf{I})\mathbf{v} = 0, \tag{A.3}$$

where $\eta = \tanh\pi\delta$. For a nontrival solution of $\mathbf{v}$, we have

$$\|\boldsymbol{\beta} + \mathrm{i}\eta\mathbf{I}\| = 0. \tag{A.4}$$

Solving the eigenvalue problem (A.3), we have the three eigenvalues, $\eta_1 = \lambda$, $\eta_2 = -\lambda$ and $\eta_3 = 0$, and the associated eigenvectors, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. In obtaining the eigenvalues, (8) has been used. A general expression for the singular solution may be written as

$$\boldsymbol{\chi}(z) = \frac{1}{2\sqrt{2\pi z}}\mathbf{V}\mathbf{Z}(z^{\mathrm{i}\varepsilon}, z^{-\mathrm{i}\varepsilon})\mathbf{V}^{-1}\mathbf{k}, \tag{A.5}$$

where $\mathbf{V} = (\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3)$ and $\mathbf{Z}(\zeta_1, \zeta_2) = \mathrm{diag}(\zeta_1 \quad \zeta_2 \quad 1)$. Defining a matrix function $\mathbf{Y}(\zeta_1, \zeta_2)$ as $\mathbf{Y}(\zeta_1, \zeta_2) = \mathbf{V}\mathbf{Z}(\zeta_1, \zeta_2)\mathbf{V}^{-1}$, it can be shown that

$$\mathbf{Y}(\zeta_1, \zeta_2) = \mathbf{I} + \tfrac{1}{2}(\zeta_1 - \zeta_2)\mathbf{X}_1 + \left\{\tfrac{1}{2}(\zeta_1 + \zeta_2) - 1\right\}\mathbf{X}_2, \tag{A.6}$$

where $\mathbf{I}_1 = \mathrm{diag}(1 \quad -1 \quad 0)$, $\mathbf{X}_1 = \mathbf{V}\mathbf{I}_1\mathbf{V}^{-1}$ and $\mathbf{X}_2 = \mathbf{V}\mathbf{I}_1^2\mathbf{V}^{-1}$. Making use of the following relations:

$$\boldsymbol{\beta}\mathbf{V} = -\mathrm{i}\mathbf{V}\boldsymbol{\Lambda}, \qquad \boldsymbol{\Lambda} = \lambda\mathbf{I}_1, \tag{A.7}$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda \quad -\lambda \quad 0)$, it can be shown that

$$\mathbf{X}_1 = \frac{\mathrm{i}}{\lambda}\boldsymbol{\beta}, \qquad \mathbf{X}_2 = \frac{-1}{\lambda^2}\boldsymbol{\beta}^2. \tag{A.8}$$

Substituting (A.8) into (A.6), we have the expression of $\mathbf{Y}(\zeta_1, \zeta_2)$ in (20). Thus, (A.5) is rewritten as

$$\boldsymbol{\chi}(z) = \frac{1}{2\sqrt{2\pi z}}\mathbf{Y}(z^{\mathrm{i}\varepsilon}, z^{-\mathrm{i}\varepsilon})\mathbf{k}. \tag{A.9}$$

Finally, we get (18) from (A.1) and (A.9).

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